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# On an Integral Equation Arising in Inverse Scattering

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ON AN INTEGRAL EQUATION ARISING IN INVERSE SCATTERING

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## ABSTRACT

A method of Fourier transforms and double series is applied for a solution of the following integral equation arising in inverse scattering:

$$\int_{-t}^x R(t+y)K(x,y)dy + K(x,t) + R(x+t) = 0, \quad t \leq x$$

where  $R(w)$  is a Fourier transform of the given reflection coefficient in the differential equation  $u''(k,x) + [k^2 - V(x)] u(k,x) = 0$ ,  $-\infty < x < \infty$ .

By assuming  $R(w)$  to be analytic, it is found that the unknown function can be constructed in the following forms:

$$1) K(x,y) = \sum_{n=0}^{\infty} A_n(x) y^n, \quad \text{ii) } K(x,y) = \sum_{p=0}^{\infty} \sum_{m+n=p} C_{m,n} x^m y^n$$

in which these series converge uniformly and absolutely for  $|x| < x(R)$ .  $x(R)$  depends on  $R(w)$ . Therefore, if the reflection coefficient of the above differential equation is given with suitable conditions,  $V(x)$  can be found by solving the integral equation and by the relation  $V(x) = 2 \frac{d}{dx} K(x,x)$ . Some related topics are discussed.

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## 1. Introduction

Let us consider the following differential equation,

$$\frac{d^2}{dx^2} U(k, x) + [k^2 - V(x)] U(k, x) = 0,$$

which appears in many physical problems.

If  $V(x)$  is given, we can obtain, in principle, a solution such that

$$\begin{aligned} U(k, x) &\sim e^{ikx} + r(k) e^{-ikx}, \quad x \rightarrow -\infty \\ &\quad (-\infty < x < \infty) \\ U(k, x) &\sim t(k) e^{ikx}, \quad x \rightarrow +\infty \end{aligned}$$

where  $r(k)$ ,  $t(k)$  are respectively reflection and transmission coefficients.\*

Conversely, one can obtain the ionization density from a knowledge of the reflection coefficient. The latter data are obtainable from the time delay of a pulsed radio wave transmitted from the earth and reflected back by the ionosphere. This ionization density is calculated as follows:

a) we first determine the kernel  $K(x, y)$  from the integral equation

$$(1) \quad \int_{-t}^x R(t+y) K(x, y) dy + K(x, t) + R(x+t) = 0, \quad t \leq x$$

where  $R(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} r(k) e^{-ikt} dk$ ,  $r(k)$  being a complex reflection coefficient.\*\*

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\* See [4] Appendix and [1] p. 5-7.

\*\* See [1], section 3. We assume that  $R(t)$  is analytic for  $t \geq 0$  in this report unless specified.

b) We next calculate  $V(x) = 2 \frac{d}{dx} K(x, x)$ .

c) Finally  $V(x)$  is proportional to the ionization density.

Equation (1) can be regarded as the following Fredholm integral equation

$$(2) \quad -\lambda \int_{-x}^x R(t+y) f(y) dy + f(t) + g(t) = 0 \quad (x = \text{fixed})$$

with the condition  $R(s) = 0$  for  $s < 0$ ;  $\lambda = -1$ ,  $f(y) = K(x, y)$  and  $g(t) = R(x+t)$ .

It follows that for every  $t \in [-x, x]$ ,  $f(t)$  has a solution belonging to  $L_2$  (or  $C$ ) if  $R(s) \in L_2$  (or  $C$ ), where  $L_2$  is the class of square summable functions ( $C$  being the class of continuous functions).

Evaluation of a solution of (2) is very difficult, due to the condition  $R(s) = 0$  for  $s < 0$ , by the standard Fredholm theory. It is therefore desirable to develop a new method of solving (2). From (1), after changing variables  $z = t+y$ ,  $w = x+t$ , we obtain

$$(3) \quad \int_0^w R(z) K(x, x+z-w) dz + K(x, w-x) + R(w) = 0, \quad w \geq 0$$

which can also be written as follows:

$$(4) \quad \int_0^w R(w-u) K(x, x-u) du + K(x, w-x) + R(w) = 0, \quad w \geq 0.$$

Applying a Fourier transform to (4), for the intervals  $(0, \infty)$  and  $(-\infty, 0)$ , we obtain the following equations, respectively:

$$(1) \quad r_+(\alpha) \int_{-\infty}^x K(x, y) e^{-i\alpha y} dy + \int_{-x}^{\infty} K(x, y) e^{i\alpha y} dy + e^{-i\alpha x} r_+(\alpha) = 0 \quad *$$

for  $I_m(\alpha) \gg 0$

---

\*See Appendix, p. 26.  $I_m(\alpha)$  means the imaginary part of  $\alpha$ ;  $A \gg 0$  means  $A$  is positive and very large. Similarly  $A \ll 0$  means  $A$  is negative and very large.



$$(ii) \quad -r_-(\alpha) \int_0^\infty K(x,y) e^{-i\alpha y} dy + \int_{-\infty}^{-x} K(x,y) e^{i\alpha y} dy + e^{-i\alpha x} r_-(\alpha) = 0$$

$$\text{for } I_m(\alpha) \ll 0$$

$$\text{where } r_+(\alpha) = \int_0^\infty R(t) e^{i\alpha t} dt = r(\alpha), \quad r_-(\alpha) = \int_{-\infty}^0 R(t) e^{i\alpha t} dt$$

Here, we have extended the definition of  $R(t)$  for  $t \leq 0$ .<sup>\*\*</sup>

Substituting  $-\alpha$  for  $\alpha$  into i) and ii), we obtain, respectively, the following:

$$(iii) \quad r_+(-\alpha) \int_{-\infty}^x K(x,y) e^{i\alpha y} dy + \int_x^\infty K(x,y) e^{-i\alpha y} dy + e^{i\alpha x} r_+(-\alpha) = 0$$

$$\text{for } I_m(\alpha) \ll 0$$

$$(iv) \quad -r_-(-\alpha) \int_x^\infty K(x,y) e^{i\alpha y} dy + \int_{-\infty}^{-x} K(x,y) e^{-i\alpha y} dy + e^{i\alpha x} r_-(-\alpha) = 0$$

$$\text{for } I_m(\alpha) \gg 0.$$

The sets of equations (i,iv) and (ii,iii) cannot be solved uniquely unless we have more conditions on  $K(x,y)$  since there are four unknowns but only two equations, respectively. Nevertheless, when  $x = 0$ , they are solvable for  $K(0,y)$  and we obtain the following equations from (i-iv):

$$(i') \quad r_+(\alpha) \int_{-\infty}^0 K(0,y) e^{-i\alpha y} dy + \int_0^\infty K(0,y) e^{i\alpha y} dy + r_+(\alpha) = 0, \quad I_m(\alpha) \gg 0$$

$$(ii') \quad -r_-(\alpha) \int_0^\infty K(0,y) e^{-i\alpha y} dy + \int_{-\infty}^0 K(0,y) e^{i\alpha y} dy + r_-(\alpha) = 0, \quad I_m(\alpha) \ll 0$$

$$(iii') \quad r_+(-\alpha) \int_{-\infty}^0 K(0,y) e^{i\alpha y} dy + \int_0^\infty K(0,y) e^{-i\alpha y} dy + r_+(-\alpha) = 0, \quad I_m(\alpha) \ll 0$$

<sup>\*\*</sup>

We take the analytical continuation of  $R(t)$  as its extended definition for  $t \leq 0$ , in the above integral.

$$(iv') \quad -r_-(-\alpha) \int_0^\infty K(0, y) e^{i\alpha y} dy + \int_{-\infty}^0 K(0, y) e^{-i\alpha y} dy + r_-(-\alpha) = 0, \\ I_m(\alpha) \gg 0.$$

From (i') and (iv') we obtain

$$(5) \quad \hat{K}_+(0, \alpha) = \int_0^\infty K(0, y) e^{i\alpha y} dy = \left| \begin{array}{cc} r_+(\alpha), r_+(\alpha) \\ 1, r_-(-\alpha) \end{array} \right| \bigg/ \left| \begin{array}{cc} 1, r_+(\alpha) \\ -r_-(-\alpha), 1 \end{array} \right| \\ = r_+(\alpha) [r_-(-\alpha) - 1] \bigg/ [1 + r_+(\alpha) r_-(-\alpha)].$$

Similarly, from (ii') and (iii'), we obtain

$$(6) \quad \hat{K}_-(0, \alpha) = \int_{-\infty}^0 K(0, y) e^{i\alpha y} dy = -r_-(-\alpha) [1 + r_+(-\alpha)] \bigg/ [1 + r_+(-\alpha) r_-(-\alpha)].$$

It follows that

$$(7) \quad K(0, y) = \frac{1}{2\pi} \int_{ic-\infty}^{ic+\infty} \hat{K}_+(0, \alpha) e^{-i\alpha y} d\alpha + \frac{1}{2\pi} \int_{id-\infty}^{id+\infty} \hat{K}_-(0, \alpha) e^{-i\alpha y} d\alpha$$

for  $c \gg 0$  and  $d \ll 0$ .

According to Fredholm's theory, the solution of (2) can be written as follows:

$$f(t) = -g(t) - \int_{-x}^x K_R(t, y; \lambda) g(y) dy$$

where

$$K_R(t, y) = D(t, y; \lambda) / D(\lambda) \quad \text{if } D(\lambda) \neq 0, \\ D(\lambda) = 1 + \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \int_{-x}^x \dots \int_{-x}^x R \left( \begin{array}{c} s_1, \dots, s_n \\ s_1, \dots, s_n \end{array} \right) ds_1 \dots ds_n, \\ D(t, y; \lambda) = 1 + \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \int_{-x}^x \dots \int_{-x}^x R \left( \begin{array}{c} t, s_1, \dots, s_n \\ y, s_1, \dots, s_n \end{array} \right) ds_1 \dots ds_n,$$

and

$$R \begin{pmatrix} x_1, \dots, x_n \\ y_1, \dots, y_n \end{pmatrix} = \begin{vmatrix} R(x_1+y_1), \dots, R(x_1+y_n) \\ \vdots \\ R(x_n+y_1), \dots, R(x_n+y_n) \end{vmatrix}.$$

It follows that the solution of (1)

$$K(x, t) = -R(x+t) - \frac{1}{D(-1)} \int_{-t}^x D(t, y; -1) R(x+y) dy$$

is a function analytic with respect to  $t$  and meromorphic with respect to  $x$  if  $R(w)$  is analytic with respect to  $w$ .

If  $|\lambda_0(x)| > 1$  then  $D(-1) \neq 0$ , where  $\lambda_0(x)$  is the first eigenvalue of (2), and this is always true if  $x$  is sufficiently small since  $|\lambda_0(x)|^2$  is greater than  $\left\{ \int_0^x \int_{-y}^x |R(t+y)|^2 dt dy \right\}^{-1}$ .

It follows that  $K(x, t)$  can always be expanded in a power series in  $x$  and  $t$  for small  $x$ . And  $K(x, y)$  can also be expanded in a power series in  $x$  with coefficients as functions of  $y$ . In the following section, we shall discuss the two cases in which we could find the solution of (1) by a method of Fourier transforms.

2. The case where  $K(x, y) = \sum_{n=0}^{\infty} A_n(y) x^n$

As in Section 1, we can assume that  $K(x, y)$  can be expanded in power series of  $x$ :  $K(x, y) = \sum_{n=0}^{\infty} A_n(y) x^n$  which is uniformly and absolutely convergent for small  $x$ , e.g. we can take any value of  $x$

which is less than  $x_0 = \sup \left\{ a \left| \int_0^a \int_{-y}^a |R(t+y)|^2 dt dy < 1 \right. \right\}$ .

Theorem 1:

Let  $R(t)$  be a given analytic function, then the integral equation (1) has a solution  $K(x, y) = \sum_{n=0}^{\infty} A_n(y) x^n$  for  $|x| < x_0$

where  $A_n(y)$  can be obtained from the following recursion formula:

$$(8) \quad \begin{aligned} A_n(y) = & \frac{1}{2\pi} \int_{ic-\infty}^{ic+\infty} \frac{r_+^0(\alpha) r_+^n(-\alpha) - r_+^n(\alpha)}{1 + r_+^0(\alpha) r_-^0(-\alpha)} e^{-iy\alpha} d\alpha \quad (c \gg 0) \\ & - \frac{1}{2\pi} \int_{id-\infty}^{id+\infty} \frac{r_-^0(\alpha) r_+^n(-\alpha) + r_-^n(\alpha)}{1 + r_+^0(-\alpha) r_-^0(\alpha)} e^{-iy\alpha} d\alpha \quad (d \ll 0) \end{aligned}$$

where

$$(9) \quad r_+^n(\alpha) = \int_0^\infty R^n(0, t) e^{i\alpha t} dt, \quad I_m(\alpha) \gg 0 \quad (n \geq 0)$$

$$(10) \quad r_-^n(\alpha) = \int_{-\infty}^0 R^n(0, t) e^{i\alpha t} dt, \quad I_m(\alpha) \ll 0$$

$$(11) \quad R^n(x, t) = \frac{1}{x} \left\{ \int_{-t}^x R(t+y) H_{n-1}(0, y) dy + H_{n-1}(0, t) + R^{n-1}(x, t) \right\}$$

$$(12) \quad H_n(x, t) = \frac{1}{x} \left\{ H_{n-1}(x, y) - H_{n-1}(0, y) \right\} \quad (n \geq 1)$$

$$(13) \quad R^0(x, y) = R(x+y)$$

$$(14) \quad H_0(x, y) = K(x, y).$$

It is known that when  $n=0$ ,  $A_0(y) = K(0, y)$  and formula (8) becomes

identical with (7); when  $n=1$ ,  $A_1(y) = H_1(0,y) = \left. \frac{\partial}{\partial x} K(x,y) \right|_{x=0}$ , and

so on, and in general  $A_n(y) = H_n(0,y) = \left. \left( \frac{\partial}{\partial x} \right)^n K(x,y) \right|_{x=0}$ .

### Proof of Theorem 1:

When  $n=0$ , we have already proven the theorem in Section 1.

By Mathematical Induction, suppose it has been proven for the cases  $k < n$ , then for  $k=n$ , we would have from (1)

$$(15) \quad \int_{-t}^x R(t+y) H_n(x,y) dy + H_n(x,t) + R^n(x,t) = 0.$$

This is true since, by our assumption, we already know that

$$(16) \quad \int_{-t}^x R(t+y) H_{n-1}(x,y) dy + H_{n-1}(x,t) + R^{n-1}(x,t) = 0$$

holds. It follows from (16) that

$$(17) \quad \int_{-t}^x R(t+y) \left\{ H_{n-1}(x,y) - H_{n-1}(0,y) \right\} dy + H_{n-1}(x,t) - H_{n-1}(0,t) + R^{n-1}(x,t) + \int_{-t}^x R(t+y) H_{n-1}(0,y) dy + H_{n-1}(0,t) = 0.$$

Dividing (17) by  $x$ , and using the formulas (11) and (12), we obtain the integral equation (15). Applying a method similar to that used in Section 1, we obtain

$$(I) \quad r_+^0(\alpha) \int_{-\infty}^x H_n(x,y) e^{-i\alpha y} dy + \int_{-x}^{\infty} H_n(x,y) e^{i\alpha y} dy + r_+^n(x,\alpha) = 0$$

$$\text{Im}(\alpha) \gg 0$$

$$(II) \quad -r_-^0(\alpha) \int_x^\infty H_n(x,y) e^{-i\alpha y} dy + \int_{-\infty}^{-x} H_n(x,y) e^{i\alpha y} dy + r_-^n(x, \alpha) = 0$$

$$\operatorname{Im}(\alpha) \ll 0$$

where

$$r_+^n(x, \alpha) = \int_{-x}^\infty R^n(x, y) e^{i\alpha y} dy$$

$$r_-^n(x, \alpha) = \int_{-\infty}^{-x} R^n(x, y) e^{i\alpha y} dy \quad .$$

Putting  $x=0$ , and using the same procedure as in Section 1, we obtain formula (8), where  $r_+^n(\alpha) \equiv r_+^n(0, \alpha)$ ,  $r_-^n(\alpha) \equiv r_-^n(0, \alpha)$ .

3. The case where  $K(x, y) = \sum_{n=0}^{\infty} \sum_{m+n=p} C_{m,n} x^m y^n$

Theorem 2:

$$\text{Let } R(w) = \sum_{n=0}^{\infty} \gamma_n w^n \text{ be given, then the integral equation (1)}$$

has a solution  $K(x, y) = \sum_{p=0}^{\infty} \sum_{m+n=p} C_{m,n} x^m y^n$  for  $|y| \leq |x| \leq x_0$ , where

$$C_{m,n} \quad \text{can be determined from } \gamma_k \quad \text{by the following} \\ (m, n=0, 1, 2, \dots) \quad (k=0, 1, 2, \dots)$$

recursion formula:

$$\left( \text{Here, } x_0 = \sup \left\{ x \mid \int_0^x \int_{-y}^x |R(t+y)|^2 dt dy < 1 \right\} \right) \\ (18) \quad \delta_{0,q} \gamma_l + \sum_{k=0}^{l-1} \sum_{m=0}^q (-1)^k \binom{k+q-m}{q-m} \frac{C_{m,k+q-m} \gamma_{l-k-1}}{l \binom{l-1}{k}} + \sum_{m=0}^q (-1)^{q-m} C_{m,l+q-m} \binom{l+q-m}{q-m} = 0$$

for  $q, l = 0, 1, 2, 3, \dots$  (Here, we use the convention  $\sum_{k=0}^{-1} f_k = 0$ ,  $\delta_{ij}$

being Kronecker's delta).

The procedure to obtain  $C_{m,n}$  from (18) is as follows:

Putting  $q=0$ ,  $l=0,1,2,3,\dots$ , we can obtain  $C_{o,m}$  ( $m=0,1,2,\dots$ ), step by step; then we obtain  $C_{1,o}$  from  $C_{o,o}$  and  $C_{o,1}$ , and so on. We obtain  $C_{1,m}$  from  $C_{o,p}$  ( $0 \leq p \leq m+1$ ) and  $C_{1,q}$  ( $0 \leq q \leq m-1$ ). Similarly, we obtain  $C_{m,n}$  from  $C_{p,q}$  ( $0 \leq p \leq m-1$ ,  $0 \leq q \leq m+1$ ) and  $C_{m,l}$  ( $0 \leq l \leq n-1$ ).

### Proof of Theorem 2:

Since  $K(x,y) = \sum \sum C_{m,n} x^m y^n$ , we obtain

$$\begin{aligned} K(x, x-u) &= \sum_{p=0}^{\infty} \sum_{m+n=p} C_{m,n} x^m (x-u)^n \\ &= \sum_{p=0}^{\infty} \sum_{m=0}^p C_{m,p-m} \sum_{q=m}^p (-1)^{p-q} \binom{p-m}{q-m} u^{p-q} x^q \\ &= \sum_{p=0}^{\infty} \sum_{q=0}^p \left( \sum_{m=0}^q C_{m,p-m} (-1)^{p-q} \binom{p-m}{q-m} u^{p-q} \right) x^q \\ &= \sum_{q=0}^{\infty} \left( \sum_{p=q}^{\infty} \sum_{m=0}^q C_{m,p-m} (-1)^{p-q} \binom{p-m}{q-m} u^{p-q} \right) x^q. \end{aligned}$$

Similarly

$$\begin{aligned} (19) \quad K(x, w-x) &= \sum_{p=0}^{\infty} \sum_{m+n=p} C_{m,n} x^m (w-x)^n \\ &= \sum_{q=0}^{\infty} \left( \sum_{p=q}^{\infty} \sum_{m=0}^q C_{m,p-m} (-1)^{q-m} \binom{p-m}{q-m} w^{p-q} \right) x^q, \end{aligned}$$

and

$$\begin{aligned} (20) \quad \int_0^w R(w-u) K(x, x-u) du &= \int_0^w R(w-u) \left\{ \sum_{q=0}^{\infty} \left( \sum_{p=q}^{\infty} \sum_{m=0}^q C_{m,p-m} (-1)^{p-q} \binom{p-m}{q-m} u^{p-q} \right) u^{p-q} \right\} du \\ &= \sum_{q=0}^{\infty} \left\{ \int_0^w R(w-u) \left[ \sum_{p=q}^{\infty} \sum_{m=0}^q C_{m,p-m} (-1)^{p-q} \binom{p-m}{q-m} u^{p-q} \right] du \right\} x^q. \end{aligned}$$

Substituting (19) and (20) into (4) and arranging in powers of  $x$ , we obtain from the coefficient of  $q^{\text{th}}$  power the following:

$$(21) \quad \int_0^w R(w-u) \left\{ \sum_{p=q}^{\infty} \sum_{m=0}^q C_{m,p-m} (-1)^{p-q} \binom{p-m}{q-m} u^{p-q} \right\} du$$

$$+ \sum_{p=q}^{\infty} \sum_{m=0}^q C_{m,p-m} (-1)^{q-m} \binom{p-m}{q-m} w^{p-q} + \delta_{0,q} R(w) = 0.$$

Also, we have

$$\begin{aligned} & \int_0^w R(w-u) \left\{ \sum_{p=q}^{\infty} \sum_{m=0}^q C_{m,p-m} (-1)^{p-q} \binom{p-m}{q-m} u^{p-q} \right\} du \\ &= \sum_{p=q}^{\infty} \sum_{m=0}^q C_{m,p-m} (-1)^{p-q} \binom{p-m}{q-m} \int_0^w u^{p-q} \sum_{n=0}^{\infty} \gamma_n (w-u)^n du \\ & \quad (\text{put } \ell = n+p-q) \\ &= \sum_{p=q}^{\infty} \sum_{m=0}^q C_{m,p-m} (-1)^{p-q} \binom{p-m}{q-m} \sum_{n=0}^{\infty} \gamma_n \sum_{s=0}^n \binom{n}{s} (-1)^{n-s} \int_0^w u^{n-s+p-q} du \cdot w^s \\ &= \sum_{p=q}^{\infty} \sum_{m=0}^q C_{m,p-m} (-1)^{p-q} \binom{p-m}{q-m} \sum_{\ell=p-q}^{\infty} \gamma_{\ell+q-p} \sum_{s=0}^{\ell+q-p} \binom{\ell+q-p}{s} \frac{(-1)^{\ell+q+p+s} w^{\ell+1}}{\ell-s-1} \\ & \quad (\text{put } k=p-q) \\ &= \sum_{\ell=0}^{\infty} \left\{ \sum_{k=0}^{\ell} \sum_{m=0}^q C_{m,k+q-m} (-1)^k \binom{k+q-m}{q-m} \cdot \frac{\gamma_{\ell-k}}{(\ell+1) \binom{\ell}{k}} \right\} w^{\ell+1}. \end{aligned}$$

$$(\text{Here we use } (*)) \dots \sum_{s=0}^{\ell-k} \binom{\ell-k}{s} \frac{(-1)^{\ell+k+s}}{\ell-s+1} = \frac{1}{(\ell+1) \binom{\ell}{k}} (*).$$

---

\* See Appendix



Substituting the above expression into (21) and equating the coefficient of the  $l^{\text{th}}$  power of  $w$ , we obtain (18).

#### 4. Remarks, Special Cases, and Examples.

A recursion formula equivalent to the identity (18) can be derived from equation (1) if  $K(x, y)$  satisfies some conditions such that integrations by parts are possible in equation (1). We have the following theorem:

Theorem 3: If  $\left(\frac{\partial}{\partial y}\right)^n K(x, y) = O(e^c |y|)$  for a fixed constant  $c$  and for all natural numbers  $n$ , then by integrations by parts we can obtain the following double recursion formula from equation (1):

$$(22) \quad \delta_{0, s} k_r - r! \sum_{n=r}^{s+r} (-1)^n \binom{n}{r} C_{s+r-n, n} - \sum_{m=0}^{r-1} \sum_{n=r-m-1}^{s+y-m-1} (r-m-1)! k_m \binom{n}{r-m-1} \\ \cdot C_{s+r-m-1-n, n} = 0$$

where

$$r_+(\alpha) = \sum_{n=0}^{\infty} \frac{k_n}{(i\alpha)^n}, \quad K(x, y) = \sum_{p=0}^{\infty} \sum_{m+n=p} C_{m, n} x^m y^n \quad \text{being assumed.}$$

Proof: By integrating by parts  $N$  times, we obtain

$$\int_{-\infty}^x K(x, y) e^{-i\alpha y} dy = -e^{-i\alpha x} \sum_{n=0}^{N-1} \left( \frac{\partial}{\partial y} \right)^n K(x, y) \Big|_{y=x} (i\alpha)^{-n-1} + O(\alpha^{-N-1}) \\ \int_{-x}^{\infty} K(x, y) e^{i\alpha y} dy = e^{-i\alpha x} \sum_{n=0}^{N-1} \left( \frac{\partial}{\partial y} \right)^n K(x, y) \Big|_{y=-x} (-i\alpha)^{-n-1} + O(\alpha^{-N-1}).$$

Equating the  $r^{\text{th}}$  power of  $(i\alpha)^{-1}$  in the equation (1), we obtain

$$(23) \quad \left[ k_r + (-1)^{r-1} \left( \frac{\partial}{\partial y} \right)^r K(x, y) \right]_{y=-x} - \sum_{m=0}^{r-1} k_m \left( \frac{\partial}{\partial y} \right)^{r-m-1} K(x, y) \Big|_{y=x} = 0 \quad (r \leq N)$$

Putting  $K(x, y) = \sum_{p=0}^{\infty} \sum_{m+n=p} c_{m,n} x^m y^n$  and equating the  $s^{\text{th}}$  power of  $x$

in the identity (23), we obtain the identity (22). If we assume

$$R(w) = \sum_{n=0}^{\infty} \gamma_n w^n \text{ then } r_+(\alpha) = \sum_{n=0}^{\infty} \frac{k_n}{(i\alpha)^{n+1}} = \int_0^{\infty} R(w) e^{i\alpha w} dw$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} n! \gamma_n}{(i\alpha)^{n+1}}$$

i.e.

$$(24) \quad k_n = (-1)^{n+1} n! \gamma_n.$$

Substituting the identity (24) for  $k_n$  into the identity (22) and dividing by  $r!$  in the latter, we can easily obtain the identity (18) if we make a suitable change of variables.

In Theorem 2 (or Theorem 3), computation of  $C_{m,n}$  is sometimes tedious; e.g.  $C_{m,n} (m+n \leq 5)$  in terms of  $\gamma_i$  are as follows:

[We could obtain  $C_{m,n}$  in terms of  $k_i$  if we used the identity (24).]

$$C_{0,0} = -\gamma_0$$

$$C_{0,1} = -\gamma_1 + \gamma_0^2$$

$$C_{0,2} = -\gamma_2 + \frac{1}{2}\gamma_0^3$$

$$C_{0,3} = -\gamma_3 + \frac{2}{3}\gamma_2\gamma_0 - \frac{1}{6}\gamma_1^2 + \frac{1}{6}\gamma_1\gamma_0^2 - \frac{1}{6}\gamma_0^4$$

$$C_{0,4} = -\gamma_4 + \frac{1}{4}\gamma_2\gamma_0^2 - \frac{1}{24}\gamma_1^2\gamma_0 - \frac{1}{24}\gamma_0^5$$

$$C_{0,5} = -\gamma_5 + \frac{2}{5}\gamma_4\gamma_0 - \frac{1}{10}\gamma_3\gamma_1 + \frac{1}{20}\gamma_3\gamma_0^2 + \frac{1}{30}\gamma_2^2 + \frac{1}{30}\gamma_2\gamma_1\gamma_0 - \frac{1}{15}\gamma_2\gamma_0^3$$

$$- \frac{1}{120}\gamma_1^3 + \frac{1}{60}\gamma_1^2\gamma_0^2 - \frac{1}{120}\gamma_1\gamma_0^4 + \frac{1}{120}\gamma_0^6$$

$$c_{1,0} = c_{0,1} = -\gamma_1 + \gamma_0^2$$

$$c_{1,1} = -2\gamma_2 + 2\gamma_1\gamma_0 - \gamma_0^3$$

$$c_{1,2} = -3\gamma_3 + \frac{1}{2}\gamma_1^2 + \frac{1}{2}\gamma_1\gamma_0^2 - \frac{1}{2}\gamma_0^4$$

$$c_{1,3} = -4\gamma_4 + 2\gamma_3\gamma_0 - \frac{1}{3}\gamma_2\gamma_0^2 + \frac{1}{6}\gamma_1^2\gamma_0 - \frac{1}{3}\gamma_1\gamma_0^3 + \frac{1}{6}\gamma_0^5$$

$$c_{1,4} = -5\gamma_5 + \frac{1}{2}\gamma_3\gamma_1 + \frac{1}{4}\gamma_3\gamma_0^2 - \frac{1}{6}\gamma_2^2 + \frac{1}{6}\gamma_2\gamma_1\gamma_0 - \frac{1}{6}\gamma_2\gamma_0^2 - \frac{1}{24}\gamma_1^3 \\ - \frac{1}{24}\gamma_1^4 + \frac{1}{24}\gamma_0^6$$

$$c_{2,0} = -\gamma_2 + 2\gamma_1\gamma_0 - \frac{3}{2}\gamma_0^3$$

$$c_{2,1} = -3\gamma_3 + 2\gamma_2\gamma_0 + \frac{3}{2}\gamma_1^2 - \frac{7}{2}\gamma_1\gamma_0^2 + \frac{3}{2}\gamma_0^4$$

$$c_{2,2} = -6\gamma_4 + 2\gamma_2\gamma_1 - \frac{1}{2}\gamma_2\gamma_0^2 - \frac{1}{4}\gamma_1^2\gamma_0 - \gamma_1\gamma_0^3 + \frac{3}{4}\gamma_0^5$$

$$c_{2,3} = -10\gamma_5 + 4\gamma_4\gamma_0 + \gamma_3\gamma_1 - \frac{3}{2}\gamma_3\gamma_0^2 + \frac{1}{3}\gamma_2^2 - \gamma_2\gamma_1\gamma_0 + \frac{2}{3}\gamma_2\gamma_0^3 \\ + \frac{1}{4}\gamma_1^3 - \frac{1}{2}\gamma_1^2\gamma_0^2 + \frac{7}{12}\gamma_1\gamma_0^4 - \frac{1}{4}\gamma_0^6$$

$$c_{3,0} = -\gamma_3 + \frac{8}{3}\gamma_2\gamma_0 + \frac{5}{6}\gamma_1^2 - \frac{23}{6}\gamma_1\gamma_0^2 + \frac{11}{6}\gamma_0^4$$

$$c_{3,1} = -4\gamma_4 + 2\gamma_3\gamma_0 + 4\gamma_2\gamma_1 - \frac{13}{3}\gamma_2\gamma_0^2 - \frac{23}{6}\gamma_1^2\gamma_0 + \frac{17}{3}\gamma_1\gamma_0^3 - \frac{11}{6}\gamma_0^5$$

$$c_{3,2} = -10\gamma_5 + 3\gamma_3\gamma_1 - \frac{3}{2}\gamma_3\gamma_0^2 + \frac{7}{3}\gamma_2^2 - \frac{7}{3}\gamma_2\gamma_1\gamma_0 - \frac{1}{3}\gamma_2\gamma_0^3 - \frac{5}{12}\gamma_1^3 \\ + \frac{23}{12}\gamma_1\gamma_0^4 - \frac{11}{12}\gamma_0^6$$

$$c_{4,0} = -\gamma_4 + 4\gamma_3\gamma_0 + 2\gamma_2\gamma_1 - \frac{53}{12}\gamma_2\gamma_0^2 - \frac{27}{8}\gamma_1^2\gamma_0 + \frac{19}{3}\gamma_1\gamma_0^3 - \frac{19}{8}\gamma_0^5$$

$$c_{4,1} = -5\gamma_5 + 2\gamma_4\gamma_0 + \frac{11}{2}\gamma_3\gamma_1 - \frac{23}{4}\gamma_3\gamma_0^2 + \frac{17}{6}\gamma_2^2 - \frac{55}{6}\gamma_2\gamma_1\gamma_0 \\ + \frac{19}{3}\gamma_2\gamma_0^3 - \frac{41}{24}\gamma_1^3 + \frac{35}{4}\gamma_1^2\gamma_0 - \frac{209}{24}\gamma_1\gamma_0^4 + \frac{19}{8}\gamma_0^6$$

$$c_{5,0} = -\gamma_5 + \frac{32}{5}\gamma_4\gamma_0 + \frac{29}{10}\gamma_3\gamma_1 - \frac{119}{20}\gamma_3\gamma_0^2 + \frac{31}{30}\gamma_2^2 - \frac{239}{30}\gamma_2\gamma_1\gamma_0 \\ + \frac{223}{30}\gamma_2\gamma_0^3 - \frac{121}{120}\gamma_1^3 + \frac{124}{15}\gamma_1^2\gamma_0 - \frac{1201}{120}\gamma_1\gamma_0^4 + \frac{361}{120}\gamma_0^6.$$

$C_{q,l}$  can also be written in determinant form as follows:

$$(25) \quad C_{q,l} = (-1)^{l-1} \begin{vmatrix} q^f_0 & 1 & 0 & 0 & \dots & 0 \\ q^f_1 & \gamma_0 & 1 & 0 & \dots & 0 \\ q^f_2 & \frac{\gamma_1}{2} & \frac{-\gamma_0}{2} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ q^f_l & \frac{\gamma_l}{l \binom{l}{0}} & \frac{-\gamma_{l-1}}{l \binom{l-1}{1}} & \frac{\gamma_{l-2}}{l \binom{l-1}{2}} & \dots & \frac{(-1)^{l-1} \gamma_0}{l \binom{l-1}{l-1}} \end{vmatrix}$$

where

$$q^f_l = \delta_{0,q} \gamma_l + (-1)^q \sum_{m=0}^{q-1} (-1)^m \binom{l+q-m}{q-m} C_{m,l+q-m} + \sum_{k=0}^{l-1} \frac{(-1)^k \gamma_{l-k-1}}{l \binom{l-1}{k}} \\ \cdot \sum_{m=0}^{q-1} \binom{k+q-m}{q-m} C_{m,k+q-m} \quad (q, l=0, 1, 2, 3, \dots).$$

The formula (25) can be derived from the following formula (26) which is a different expression of (18), by applying Cramer's Rule for linear equations and by regarding  $C_{q,k}$  ( $0 \leq k \leq l$ ) as unknowns.

$$(26) \quad C_{q,t} + \sum_{k=0}^{t-1} (-1)^k \frac{\gamma_{t-k-1}}{t \binom{t-1}{k}} C_{q,k} + q^f_t = 0 \quad (0 \leq t \leq l).$$

It is known that we can obtain  $C_{0,m}$  easily by calculating (25); nevertheless it is convenient to calculate  $C_{p,q}$  for  $p \geq 1$  by using (16).

The integral equations (3) and (4) are of Volterra type if we regard  $K(x,y)$  as a given kernel and  $R(x)$  as unknown; therefore, the solution of (3) and (4) [as well as (1)] can be said to be an Inverse Problem of the Volterra equation. For a given  $K(x,w-x)$  in (3) [and in (4)], solving for  $R(w)$ , we obtain a solution  $R(w)$  which in general not only depends on  $w$  but also depends on  $x$ . Consequently, the condition that  $R(w)$  is independent of  $x$  is rather strong. It implies that functions  $K(x,y)$  have a particular form. We shall discuss this situation in the following special cases:

(a) When  $K(x,y) = A(x)B(y)$ , we have from the identities (23) and (24), the following:

$$(a,1) \quad \gamma_0 = -k_0 = -K(x,-x) = -A(x)B(-x)$$

$$(a,2) \quad \gamma_1 = k_1 = -\gamma_0 A(x)B(x) - A(x)B'(-x)$$

$$(a,3) \quad \gamma_2 = \frac{-1}{2} k_2 = -\frac{1}{2} \gamma_1 A(x)B(x) + \frac{1}{2} \gamma_0 A(x)B'(x) - \frac{1}{2} A(x)B''(-x).$$

From (a,1) and (a,2) we obtain, by eliminating  $A(x)$ ,

$$(a,4) \quad \gamma_1 B(-x) = \gamma_0^2 B(x) + \gamma_0 B'(-x).$$

In changing the sign of  $x$ , we have

$$(a,5) \quad \gamma_1 B(x) = \gamma_0^2 B(-x) + \gamma_0 B'(x).$$

By adding (a,4) and (a,5) and by subtracting (a,4) from (a,5) we obtain, respectively:

$$(a,6) \quad \gamma_0 \left\{ B'(x) + B'(-x) \right\} = -(\gamma_0^2 - \gamma_1^2) \left\{ B(x) + B(-x) \right\}$$

$$(a,7) \quad \gamma_0 \left\{ B'(x) - B'(-x) \right\} = (\gamma_0^2 + \gamma_1^2) \left\{ B(x) - B(-x) \right\} .$$

$$\text{By putting } G(x) = \frac{1}{2} \left\{ B(x) + B(-x) \right\}, \quad F(x) = \frac{1}{2} \left\{ B(x) - B(-x) \right\} ,$$

and differentiating (a,6) and (a,7), we obtain

$$(a,8) \quad \gamma_0^2 F''(x) + (\gamma_0^4 - \gamma_1^2) F(x) = 0$$

$$(a,9) \quad \gamma_0^2 G''(x) + (\gamma_0^4 - \gamma_1^2) G(x) = 0 .$$

By noticing that  $G(x)$  is even and  $F(x)$  is odd, and using (a,6) and

(a,7), we obtain ( $C = \text{constant}$ )

$$G(x) = \begin{cases} C(e^{\frac{a}{\gamma_0} x} + e^{\frac{-a}{\gamma_0} x}) & \text{if } |\gamma_1| > \gamma_0^2 \text{ and } a = \sqrt{\gamma_1^2 - \gamma_0^4} \\ C(e^{\frac{a}{\gamma_0} ix} + e^{\frac{-a}{\gamma_0} ix}) & \text{if } |\gamma_1| < \gamma_0^2 \text{ and } a = \sqrt{\gamma_0^4 - \gamma_1^2} \end{cases}$$

$$F(x) = \begin{cases} \frac{\gamma_1 - \gamma_0^2}{a} C(e^{\frac{a}{\gamma_0} x} - e^{\frac{-a}{\gamma_0} x}) & \text{if } |\gamma_1| > \gamma_0^2 \text{ and } a = \sqrt{\gamma_1^2 - \gamma_0^4} \\ -\frac{\gamma_0^2 - \gamma_1}{ai} C(e^{\frac{a}{\gamma_0} ix} - e^{\frac{-a}{\gamma_0} ix}) & \text{if } |\gamma_1| < \gamma_0^2 \text{ and } a = \sqrt{\gamma_0^4 - \gamma_1^2} . \end{cases}$$

Consequently, we obtain

$$(26) \quad K(x, y) = \frac{-\gamma_0 B(y)}{B(-x)} = \begin{cases} -\gamma_0 \frac{(\gamma_1 - \gamma_0^2) e^{\frac{a}{\gamma_0} y} + (\gamma_1 + \gamma_0^2) e^{\frac{-a}{\gamma_0} y}}{(\gamma_1 - \gamma_0^2) e^{\frac{a}{\gamma_0} x} + (\gamma_1 + \gamma_0^2) e^{\frac{-a}{\gamma_0} x}} & \text{if } \begin{cases} |\gamma_1| > \gamma_0^2 \\ a = \sqrt{\gamma_1^2 - \gamma_0^4} \end{cases} \\ -\gamma_0 \frac{(\gamma_0^2 - \gamma_1) \sin(\frac{a}{\gamma_0} y) - a \cos(\frac{a}{\gamma_0} y)}{(\gamma_0^2 - \gamma_1) \sin(\frac{a}{\gamma_0} x) + a \cos(\frac{a}{\gamma_0} x)} & \text{if } \begin{cases} \gamma_0^2 > |\gamma_1| \\ a = \sqrt{\gamma_0^4 - \gamma_1^2} \end{cases} \\ -\gamma_0 & \text{if } \gamma_1 = \gamma_0^2 \\ -\gamma_0 \frac{2\gamma_0 y - 1}{2\gamma_0 x + 1} & \text{if } \gamma = -\gamma_0^2 \end{cases}$$

since for  $\gamma_1 = \gamma_0^2$ , (a,6), (a,7) become, respectively,

$$F'(x) = 0 \quad \text{and} \quad G'(x) = \gamma_0 F(x) .$$

It follows that  $F(x) = C = 0$  because  $F(x)$  is odd and hence  $G(x) = 0$  - similarly for the case  $\gamma_1 = -\gamma_0^2$ . If  $\gamma_0 = 0$ , there is only the trivial solution  $K(x,y) \equiv 0$ . By the relation (a,2), if  $\gamma_1 = 0$ , we have  $-\gamma_0 B(x) + B'(-x) = 0$  from which we can show that  $\gamma_n = 0$  for  $n > 1$ . From (a,3) we find that  $\gamma_2 = \gamma_1^2/\gamma_0$  and, in general, can show that

$$\gamma_n = \gamma_0 \left(\frac{\gamma_1}{\gamma_0}\right)^n, \quad \text{i.e.,} \quad R(x) = \gamma_0 e^{\frac{\gamma_1}{\gamma_0} x} \quad \text{Conversely, if } R(x) = \gamma_0 e^{\frac{\gamma_1}{\gamma_0} x} \quad \text{then}$$

$K(x,y)$  has exactly the form of (26); we shall show this in Example I.

(b) When  $K(x,y) = A(x)B(y) + C(x)$ , we have from the identities (23) and (24), the following:

$$(b,1) \quad -\gamma_0 = A(x)B(-x) + C(x)$$

$$(b,2) \quad -\gamma_1 = A(x)B'(-x) + \gamma_0 \left\{ A(x)B(x) + C(x) \right\}$$

$$(b,3) \quad -2\gamma_2 = A(x)B''(-x) - \gamma_0 A(x)B'(x) + \gamma_1 \left\{ A(x)B(x) + C(x) \right\} .$$

Eliminating  $A(x)$ ,  $C(x)$  from (b,1-3), we obtain

$$\begin{vmatrix} B(-x) & , & 1 & , & \gamma_0 \\ B'(-x) + \gamma_0 B(x) & , & \gamma_0 & , & \gamma_1 \\ B''(-x) - \gamma_0 B'(x) + \gamma_1 B(x) & , & \gamma_1 & , & 2\gamma_2 \end{vmatrix} = 0$$

which can be written as

$$(b,4) \quad B''(-x) - \gamma_0 B'(-x) - \frac{2\gamma_2 - \gamma_0 \gamma_1}{\gamma_1 - \gamma_0} B'(-x) - \frac{2\gamma_0 \gamma_2 - \gamma_1^2}{\gamma_1 - \gamma_0} \left\{ B(x) - B(-x) \right\} = 0$$

By changing the sign of  $x$  and putting  $K = \frac{2\gamma_2 - \gamma_0 \gamma_1}{\gamma_1 - \gamma_0}$ , we have

$$B''(x) - \gamma_0 B'(-x) - KB'(x) + (\gamma_1 - \gamma_0 K) \left\{ B(-x) - B(x) \right\} = 0$$

By a procedure similar to the one used in (a), we obtain

$$(b,5) \quad G''(x) - (\gamma_0 + K) F'(x) = 0$$

$$(b,6) \quad F''(x) + (\gamma_0 - K) G'(x) + 2(\gamma_0 K - \gamma_1) F(x) = 0$$

$$\text{where } G(x) = \frac{1}{2} \left\{ B(x) + B(-x) \right\}, \quad F(x) = \frac{1}{2} \left\{ B(x) - B(-x) \right\}.$$

Differentiating (b,5-6) and eliminating  $G''(x)$ , we have

$$H''(x) - \lambda^2 H(x) = 0$$

where

$$H(x) = F'(x), \quad -\lambda^2 = \gamma_0^2 - 2\gamma_1 + 2K\gamma_0 - K^2.$$

It follows that if  $\lambda^2 \neq 0$ , we have a solution

$$B(x) = C_1 e^{\lambda x} + C_2 e^{-\lambda x} \quad (C_1, C_2 = \text{constants}).$$

And  $A(x)$ ,  $C(x)$  can be computed from  $B(x)$  by solving simultaneous linear equations (b, 1-2) with respect to  $A(x)$  and  $C(x)$ , i.e.,



$$A(x) = \frac{\begin{vmatrix} \gamma_0 & 1 \\ \gamma_1 & \gamma_0 \end{vmatrix}}{\begin{vmatrix} -B(-x), 1 \\ -B'(x) + \gamma_0 B(x), \gamma_0 \end{vmatrix}} = \frac{\gamma_0^2 - \gamma_1}{B'(-x) - \gamma_0 \{B(x) + B(-x)\}}$$

$$C(x) = \frac{\begin{vmatrix} -B(-x) & , & \gamma_0 \\ -B'(-x) + \gamma_0 B(x), \gamma_1 \end{vmatrix}}{\begin{vmatrix} -B(-x) & , & 1 \\ -B'(-x) + \gamma_0 B(x), \gamma_0 \end{vmatrix}} = \frac{-\gamma_0 B'(-x) + \gamma_0^2 B(x) + \gamma_1 B(-x)}{B'(-x) - \gamma_0 \{B(x) + B(-x)\}}.$$

For the case  $\gamma_0 = \gamma_2 = 0$ , we have  $B(x) = C_1 \sinh \sqrt{2\gamma_1} x + C_2$ ,  
hence

$$K(x, y) = A(x)B(y) + C(x) = -\sqrt{\frac{\gamma_1}{2}} \frac{\sinh \sqrt{2\gamma_1} y + \sinh \sqrt{2\gamma_1} x}{\cosh \sqrt{2\gamma_1} x}.$$

We also can show, in this case,  $\gamma_3 = \frac{\gamma_1^2}{3!}$ , etc., ... ; we shall discuss the converse in Example II.

In the following we shall show some examples which can be solved by Theorems 1-3.

Example I. If  $R(t) = be^{-ct}$  ( $b, c$ : constants)

then

$$r_+(\alpha) = \int_0^\infty R(t)e^{i\alpha t} dt = \frac{-b}{i\alpha - c}, \quad \text{Im}(\alpha) \gg 0$$

$$r_-(\alpha) = \int_{-\infty}^0 R(t)e^{i\alpha t} dt = \frac{b}{i\alpha - c}, \quad \text{Im}(\alpha) \ll 0.$$

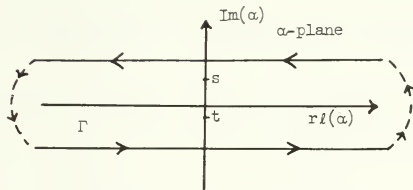
Hence

$$\hat{B}_+(\alpha) = \frac{r_+(\alpha)[r_-(-\alpha) - 1]}{1 + r_+(\alpha)r_-(-\alpha)} = \frac{(i\alpha + c + b)b}{\alpha^2 + c^2 - b^2}, \quad \text{Im}(\alpha) \gg 0$$

$$\hat{B}_-(\alpha) = \frac{-r_-(-\alpha)[r_+(\alpha) + 1]}{1 + r_+(\alpha)r_-(-\alpha)} = \frac{(i\alpha + c + b)b}{\alpha^2 + c^2 - b^2}, \quad \text{Im}(\alpha) \ll 0.$$

It follows that

$$\begin{aligned}
 B(y) &= \frac{1}{2\pi} \int_{is-\infty}^{is+\infty} \hat{B}_+(\alpha) e^{-i\alpha y} d\alpha + \frac{1}{2\pi} \int_{it-\infty}^{it+\infty} \hat{B}_-(\alpha) e^{-i\alpha y} d\alpha \\
 &\quad (s \gg 0, t \ll 0) \\
 &= \frac{1}{2\pi} \int_{\Gamma} \frac{(i\alpha+c+b)b}{\alpha^2+c^2-b^2} e^{-i\alpha y} d\alpha \left[ \begin{array}{l} d^2 = c^2 - b^2 > 0 \\ \text{similarly for other} \\ \text{cases.} \end{array} \right] \\
 &= \frac{-1}{4\pi i} \int_{\Gamma} \left\{ \frac{b}{d} \frac{d-c-b}{(\alpha-id)} + \frac{b}{d} \frac{d+c+b}{(\alpha+id)} \right\} e^{-i\alpha y} d\alpha \\
 &= \frac{-b}{2d} \left\{ (d-c-b) e^{dy} + (d+c+b) e^{-dy} \right\}
 \end{aligned}$$



$\Gamma$  is the contour consisting of two parallel lines as shown above.

Trying the form  $K(x,y) = A(x)B(y)$  for a solution of equation (1),

we obtain exactly formula (26) if we put  $\gamma_0 = b$ ,  $\gamma_1 = -bc$ .

Substituting  $K(x,y) = \frac{-b}{2d} A(x) \left\{ (d-c-b)e^{dy} + (d+c+b)e^{-dy} \right\}$  into

(1), and solving for  $A(x)$ , we obtain (26). In the above case, of course, we could assume  $K(x,y) = \sum_{n=0}^{\infty} A_n(y)x^n$  and apply Theorem 1 for

computation of  $A_n(y)$ , step by step; nevertheless, the computation is not very simple. For example, we have

$$A_0(y) = B(y) = \frac{-b}{2d} \left\{ (d-c-b)e^{dy} + (d+c+b)e^{-dy} \right\}$$

$$R^1(x, t) = \frac{1}{x} \left\{ b \int_{-t}^x e^{-c(t+y)} A_0(y) dy + A_0(t) + be^{-c(t+x)} \right\}$$

$$= \frac{b}{x} \left\{ \int_0^x e^{-c(t+y)} A_0(y) dy + e^{-c(t+x)} - e^{-ct} \right\}$$

$$\text{since } \int_{-t}^0 be^{-c(t+y)} A_0(y) dy + A_0(t) + be^{-ct} = 0.$$

It follows that

$$R^1(0, t) = b \left\{ e^{-c(t+x)} A_0(x) \right\}_{x=0} + (-c)e^{-ct}$$

$$= -b(b+c) e^{-ct}.$$

Therefore

$$r_+^{-1}(\alpha) = \int_0^\infty R^1(0, t) e^{i\alpha t} dt = \frac{b(b+c)}{i\alpha - c}, \quad \text{Im}(\alpha) \gg 0$$

$$r_-^{-1}(\alpha) = \int_{-\infty}^0 R^1(0, t) e^{i\alpha t} dt = \frac{-b(b+c)}{i\alpha - c}, \quad \text{Im}(\alpha) \ll 0.$$

Consequently

$$A_1(y) = \frac{-b(b+c)}{2\pi} \int_{\Gamma} \frac{i\alpha+b+c}{\alpha^2+c^2-b^2} e^{-iy\alpha} d\alpha \left[ \begin{array}{l} \Gamma \text{ is the same contour} \\ \text{as in p. 20.} \end{array} \right]$$

$$= \frac{b(b+c)}{2d} \left\{ (d-c-b)e^{dy} + (d+c+b)e^{-dy} \right\}.$$

Similarly we can compute  $A_2(y)$  as follows:

$$\begin{aligned}
A_1(y) &= \frac{b(b+c)}{2d} \left\{ (d-c-b)e^{dy} + (d+c+b)e^{-dy} \right\} = -(b+c)A_0(y) \\
R^2(x,t) &= \frac{1}{x} \left\{ b \int_{-t}^x e^{-c(t+y)} A_1(y) dy + A_1(t) + R^1(x,t) \right\} \\
&= \frac{1}{x} \left\{ b \int_0^x e^{-c(t+y)} A_1(y) dy + R^1(x,t) - R^1(0,t) \right\}.
\end{aligned}$$

It follows that

$$\begin{aligned}
R^2(0,t) &= be^{-ct}A_1(0) + \left. \frac{\partial}{\partial x} R^1(x,t) \right|_{x=0} \\
&= b^2(b+c)e^{-ct} + \frac{1}{2}b(b+c)^2e^{-ct} = \frac{1}{2}b(b+c)(3b+c)e^{-ct}.
\end{aligned}$$

By means of the same procedure as we used in computing  $A_1(y)$ , we obtain

$$\begin{aligned}
A_2(y) &= \frac{1}{4d} b(b+c)(3b+c) \left\{ (d-c-b)e^{dy} + (d+c+b)e^{-dy} \right\} \\
&= \frac{1}{2}(b+c)(3b+c)A_0(y).
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
A_3(y) &= -\frac{1}{3!} (b+c)^2(11b+c)A_0(y) \\
A_4(y) &= \frac{1}{4!} (b+c)^2(57b^2 + 38bc + c^2)A_0(y) \\
A_5(y) &= -\frac{1}{5!} (b+c)^3(361b^2 + 118bc + c^2)A_0(y) \quad \text{and so on.}
\end{aligned}$$

Hence

$$K(x,y) = \sum_{n=0}^{\infty} A_n(y)x^n = A_0(y) \left\{ 1 - (b+c)x + \frac{1}{2}(b+c)(3b+c)x^2 - \dots \right\}$$

from which use of the form  $K(x,y) = A(x)B(y)$  is justified.

$K(x,y)$  can be obtained from Theorem 2 (or Theorem 3) as follows:

$$\text{Since } R(t) = be^{-ct} = b \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{n!} t^n, \text{ we have } \gamma_n = \frac{(-1)^n c^n}{n!} b.$$

Using the general formula for  $C_{m,n}$  which we computed before, we obtain:

$$C_{0,0} = -b$$

$$C_{0,1} = b(c+b)$$

$$C_{0,2} = \frac{b}{2}(-c^2 + b^2) = -\frac{1}{2}bd^2$$

$$C_{0,3} = \frac{b}{6}(c^3 + bc^2 - b^2c - b^3) = \frac{bd^2}{3!}(c+b)$$

$$C_{0,4} = \frac{-b}{24}(c^4 - 2b^2c^2 + b^4) = \frac{-bd^4}{4!}$$

$$C_{0,5} = \frac{b}{120}(c^5 + bc^4 - 2bc^3 - 2b^2c^2 + b^3c + b^4) \\ = \frac{bd^2}{5}(c+b)^3$$

$$C_{2,0} = \frac{-b}{2}(c^2 + 4bcb + 3b^2) = \frac{-b}{2}(c+b)(c+3b) \\ = \frac{1}{2}(c+b)(c+3b)C_{0,0}$$

$$C_{2,1} = \frac{b}{2}(c^3 + 5bc^2 - 7b^2c + 3b^3) \\ = \frac{b}{2}(c+b)^2(c+3b) = \frac{1}{2}(c+b)(c+3b)C_{0,1}$$

$$C_{2,2} = \frac{-b}{4}(c^4 + 4bc^3 + 2b^2c^2 - 4b^3c - 3b^2) \\ = \frac{-b}{4}d^2(c+b)(c+3b) = \frac{1}{2}(c+b)(c+3b)C_{0,2}$$

$$C_{2,3} = \frac{b}{12}(c^5 + 5bc^4 + 6b^2c^3 - 2b^3c^2 - 7b^4c - 3b^5) \\ = \frac{b}{12}d^2(c+b)^2(c+3b) = \frac{1}{2}(c+b)(c+3b)C_{0,3}$$

$$C_{4,0} = \frac{-b}{24}(c^4 + 40bc^3 + 134b^2c^2 + 152b^3c + 57b^4) \\ = \frac{-b}{4!}(c+b)^2(c^2 + 38bc + 57b^2) \\ = \frac{1}{4!}(c+b)^2(c^2 + 38bc + 57b^2)C_{0,0}$$

$$C_{4,1} = \frac{b}{24}(c^5 + 41bc^4 + 174b^2c^3 + 286b^3c^2 \\ + 209b^4c + 57b^5) \\ = \frac{b}{4!}(c+b)^3(c^2 + 38bc + 57b^2) \\ = \frac{1}{4!}(c+b)^2(c^2 + 38bc + 57b^2)C_{1,0}$$

$$C_{1,0} = b(c+b) = -(c+b)C_{0,0}$$

$$\begin{aligned} C_{1,1} &= -b(c^2+2bc+b^2) + -b(c+b)^2 \\ &= -(c+b)C_{0,1} \end{aligned}$$

$$\begin{aligned} C_{1,2} &= \frac{b}{2}(c^3+bc^2-b^2c-b^3) = \frac{bd^2}{2}(c+b) \\ &= -(c+b)C_{0,2} \end{aligned}$$

$$\begin{aligned} C_{1,3} &= \frac{-b}{6}(c^4+2c^3b-2cb^3-b^4) \\ &= \frac{-bd^2}{3!}(c+b)^2 = -(c+b)C_{0,3} \end{aligned}$$

$$\begin{aligned} C_{1,4} &= \frac{b}{24}(c^5+bc^4-2b^2c^3-2b^3c^2+b^4c+b^5) \\ &= \frac{bd^4}{4!}(c+b) = -(c+b)C_{0,4} \end{aligned}$$

$$\begin{aligned} C_{3,0} &= \frac{b}{6}(c^3+13bc^2+23b^2c+11b^2) \\ &= \frac{b}{3!}(c+b)^2(c+11b) = \frac{-1}{3!}(c+b)^2(c+11b)C_{0,0} \end{aligned}$$

$$\begin{aligned} C_{3,1} &= \frac{-b}{6}(c^4+14bc^3+36b^2c^2+34b^3c+11b^4) \\ &= \frac{-b}{3!}(c+b)^3(c+11b) = \frac{-1}{3!}(c+b)^2(c+11b)C_{0,1} \end{aligned}$$

$$\begin{aligned} C_{3,2} &= \frac{b}{12}(c^5+13bc^4+22b^2c^3-2b^3c^2-23b^4c-11b^5) \\ &= \frac{bd^2}{12}(c+1)^2(c+11b) = \frac{-1}{3!}(c+b)^2(c+11b)C_{0,2} \end{aligned}$$

$$\begin{aligned} C_{5,0} &= \frac{b}{120}(c^5+121bc^4+718b^2c^3+1438b^3c^2 \\ &\quad + 1201b^4c + 361b^5) \\ &= \frac{b}{5!}(c+b)^3(c^2+118bc+361b^2) \\ &= \frac{-1}{5!}(c+b)^3(c^2+118bc+361b^2)C_{0,0} \end{aligned}$$

These coefficients  $C_{m,n}$  are exactly the coefficients of powers  $x^m y^n$  in

the expansion of  $K(x, y) = -b \frac{(d-c-b)e^{\bar{d}y} + (d+c+b)e^{-\bar{d}y}}{(d+c+b)e^{\bar{d}y} + (d-c-b)e^{-\bar{d}y}} \begin{pmatrix} \text{if } d^2 = c^2 - b^2 > 0, \\ \text{similarly for } d^2 \leq 0. \end{pmatrix}$

Example II. If  $R(t) = \sqrt{b} \sinh \sqrt{b} t = \frac{\sqrt{b}}{2} (e^{\sqrt{b}t} - e^{-\sqrt{b}t})$  then

$$K(x, y) = -\sqrt{\frac{b}{2}} \frac{\sinh \sqrt{2b} x + \sinh \sqrt{2b} y}{\cosh \sqrt{2b} x} . *$$

To show this, applying Theorem 1, we have

$$r_+(\alpha) = \int_0^\infty R(t) e^{i\alpha t} dt = \frac{-b}{\alpha^2 + b} , \quad \text{Im}(\alpha) \gg 0$$

$$r_-(\alpha) = \int_{-\infty}^0 R(t) e^{i\alpha t} dt = \frac{b}{\alpha^2 + b} , \quad \text{Im}(\alpha) \gg 0 .$$

From the formula (7), we obtain

$$\hat{K}_+(0, \alpha) = \frac{b}{\alpha^2 + 2b} , \quad \text{Im}(\alpha) \gg 0$$

$$\hat{K}_-(0, \alpha) = \frac{-b}{\alpha^2 + 2b} , \quad \text{Im}(\alpha) \ll 0 .$$

Hence

$$\begin{aligned} A_0(y) = K(0, y) &= \frac{-1}{2\pi} \int_{\Gamma} \frac{be^{-iy\alpha}}{\alpha^2 + 2b} d\alpha \left[ \begin{array}{l} \Gamma \text{ is the same contour} \\ \text{as in Example I.} \end{array} \right] \\ &= \frac{-b}{2\sqrt{2}} \frac{1}{2\pi i} \int_{\Gamma} \left\{ \frac{1}{\alpha - i\sqrt{2b}} - \frac{1}{\alpha + i\sqrt{2b}} \right\} e^{-iy\alpha} d\alpha \\ &= -\sqrt{\frac{b}{2}} \sinh \sqrt{2b} y . \end{aligned}$$

Similarly, we obtain

$$A_1(y) = -b$$

---

\* For  $b = 1$ , see [1], p. 19-21.

$$A_2(y) = b\sqrt{\frac{b}{2}} \sinh \sqrt{2b} y$$

$$A_3(y) = \frac{2}{3} b^2$$

$$A_4(y) = \frac{-5}{6} \sqrt{\frac{b}{2}} b^2 \sinh \sqrt{2b} y$$

$$A_5(y) = \frac{-8}{15} b^3 \quad \text{etc.}$$

Consequently,

$$K(x,y) = \sum_{n=0}^{\infty} A_n(y) x^n = -\sqrt{\frac{b}{2}} \sinh \sqrt{2b} y (1 - 6x + \frac{5}{6} b^2 x + \dots) \\ + (-bx + \frac{2}{3} b^2 x^3 - \frac{8}{15} b^3 x^5 + \dots)$$

from which it is reasonable for us to assume that  $K(x,y)$  has the form  $-\sqrt{\frac{b}{2}} A(x) \sinh \sqrt{2b} y + C(x)$ ; substituting the latter into (1), we obtain  $A(x) = \operatorname{sech} \sqrt{2b} x$ ,  $C(x) = -\sqrt{\frac{b}{2}} \tanh \sqrt{2b} x$ .

It also can be seen from Theorem 2 (or Theorem 3) that, in this case, we have  $\gamma_{2n} = 0$ ,  $\gamma_{2n+1} = \frac{b^{n+1}}{(2n+1)!}$ ; hence

$$\begin{array}{ccccc} c_{0,2n} = 0 & c_{1,0} = -b & c_{2,0} = 0 & c_{3,0} = \frac{2}{3} b^2 & c_{4,0} = 0 \\ c_{0,0} = -b & c_{1,1} = 0 & c_{2,1} = b^2 & c_{3,1} = 0 & c_{4,1} = \frac{-5}{6} b^3 \\ c_{0,3} = -\frac{2}{3!} b^2 & c_{1,2} = 0 & c_{2,2} = 0 & c_{3,2} = 0 & c_{4,2} = 0 \\ c_{0,5} = -\frac{2^2}{5!} b^3 & c_{1,3} = 0 & c_{2,3} = \frac{2b^3}{3} & c_{3,3} = 0 & c_{4,3} = \frac{-5}{12} b^4 \end{array}$$

from which we could also guess that  $K(x,y)$  is of the form

$$A(x)B(y) + C(x) \quad \text{with} \quad B(y) = -\sqrt{\frac{b}{2}} \sinh \sqrt{2b} y, \quad C(x) = -\sqrt{\frac{b}{2}} \tanh \sqrt{2b} x.$$

Substituting these into (1), we obtain  $A(x) = \operatorname{sech} \sqrt{2b} x$ .



From the above example, it is known that Theorem 2 is useful in determining the form of  $K(x,y)$  for a given  $R(t)$ ; similarly, for a given  $r_+(\alpha)$ , Theorem 3 is also useful in determining the form of  $K(x,y)$ . As in Example II, it is found that if  $R(t)$  is an odd function (i.e.  $\gamma_{2n} = 0$ ), then  $C_{m,n} = 0$  for  $m+n$  is even<sup>\*</sup>; and hence  $K(x,x)$  is also an odd function. Consequently,  $V(x) = 2 \frac{d}{dx} K(x,x)$  is an even function. When  $R(t)$  is even we cannot conclude very much as in the following example.

Example III If  $R(z) = e^{-cz^2}$ , then  $\gamma_{2m} = \frac{(-1)^m c^m}{m}$ ,  $\gamma_{2m+1} = 0$ .

From Theorem 2, we then obtain  $C_{p,q}$  ( $p+q \leq 5$ ) as follows:

$$C_{0,0} = -1$$

$$C_{0,1} = 1$$

$$C_{0,2} = c + \frac{1}{2}$$

$$C_{0,3} = -\frac{2}{3}c - \frac{1}{6}$$

$$C_{0,4} = -\frac{c^2}{2} - \frac{1}{4}c - \frac{1}{24}$$

$$C_{0,5} = \frac{7}{30}c^2 + \frac{1}{15}c + \frac{1}{120}$$

$$C_{2,0} = c - \frac{3}{2}$$

$$C_{2,1} = -2c + \frac{3}{2}$$

$$C_{2,2} = -3c^2 + \frac{1}{2}c + \frac{3}{4}$$

$$C_{2,3} = \frac{7}{3}c^2 - \frac{2}{3}c - \frac{1}{4}$$

$$C_{1,0} = 1$$

$$C_{1,1} = 2c - 1$$

$$C_{1,2} = -\frac{1}{2}$$

$$C_{1,3} = -2c^2 + \frac{1}{3}c + \frac{1}{6}$$

$$C_{1,4} = \frac{-1}{6}c^2 + \frac{1}{6}c + \frac{1}{24}$$

$$C_{3,1} = -\frac{8}{3}c + \frac{11}{6}$$

$$C_{3,1} = -2c^2 + \frac{13}{3}c - \frac{11}{6}$$

$$C_{3,2} = \frac{7}{3}c^2 + \frac{1}{3}c - \frac{11}{12}$$

$$C_{4,0} = -\frac{c^2}{2} + \frac{53}{12}c - \frac{19}{8}$$

$$C_{4,1} = \frac{23}{6}c^2 - \frac{19}{3}c + \frac{19}{8}$$

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<sup>\*</sup> See Appendix

$$C_{5,0} = \frac{127}{30}c^2 - \frac{223}{30}c + \frac{361}{120} \text{ and so on.}$$

And

$$K(x, x) = -1 + 2x + 2(2c-1)x^2 + \frac{16}{3} \left(c + \frac{1}{2}\right)x^3 + \left(-8c^2 + \frac{28}{3}c - \frac{10}{3}\right)x^4 \\ + \left(\frac{64}{5}c^2 - \frac{208}{15}c + \frac{64}{15}\right)x^5 + \dots$$

#### Example IV

If  $R(z) = ze^{-cz^2}$ , then  $\gamma_{2m} = 0$ ,  $\gamma_{2m+1} = \frac{(-1)^m c^m}{m!}$ .

And we obtain  $C_{m,n} = 0$  if  $m+n$  is even.

$$C_{0,1} = -1, \quad C_{1,0} = -1, \quad C_{2,1} = 3c + \frac{3}{2} \\ C_{0,3} = c - \frac{1}{6}, \quad C_{1,2} = 3c + \frac{1}{2}, \quad C_{2,3} = -5c^2 - c + \frac{1}{4} \\ C_{0,5} = \frac{-c^2}{2} + \frac{c}{10} - \frac{1}{120}, \quad C_{1,4} = -\frac{5c^2}{2} - \frac{c}{2} - \frac{1}{24}, \quad C_{3,0} = -c + \frac{5}{6}$$

$$C_{3,2} = -5c^2 - 3c - \frac{5}{12}, \quad C_{4,1} = -\frac{5c^2}{2} - \frac{11c}{2} - \frac{41}{24}, \quad C_{5,0} = -\frac{c^2}{2} - \frac{29}{10}c - \frac{121}{120};$$

hence  $K(x, x) = -2x + \left(6c + \frac{16}{3}\right)x^3 - \left(16c^2 + \frac{64}{5}c + \frac{44}{15}\right)x^5 + \dots$

and  $V(x) = 2 \frac{d}{dx} K(x, x) = -4 + 6\left(6c + \frac{16}{3}\right)x^2 - 10\left(16c^2 + \frac{64}{5}c + \frac{44}{15}\right)x^4 + \dots$

# APPENDIX

(i) holds since

$$\begin{aligned} \int_0^\infty \int_0^w R(w-u)K(x, x-u)du \, e^{i\alpha w} dw &= \int_0^\infty K(x, x-u)e^{i\alpha u} du \int_u^\infty R(w-u)e^{i\alpha(w-u)} dw \\ &= r_+(\alpha) \int_0^\infty K(x, x-u)e^{i\alpha u} du = r_+(\alpha) \int_{-\infty}^x K(x, y)e^{-i\alpha y} dy \cdot e^{i\alpha x} \\ \int_0^\infty K(x, w-x)e^{i\alpha w} dw &= \int_x^\infty K(x, y)e^{i\alpha y} dy \cdot e^{i\alpha x} . \end{aligned}$$

Similarly, (ii) holds since

$$\begin{aligned} \int_{-\infty}^0 \int_0^w R(w-u)K(x, x-u)du \, e^{i\alpha w} dw &= - \int_{-\infty}^0 K(x, x-u)e^{i\alpha u} du - \int_{-\infty}^u R(w-u)e^{i\alpha(w-u)} dw \\ &= -r_-(\alpha) \int_{-\infty}^0 K(x, x-u)e^{i\alpha u} du = -r_-(\alpha) \int_x^\infty K(x, y)e^{-i\alpha y} dy \cdot e^{i\alpha x} \\ \int_{-\infty}^0 K(x, w-x)e^{i\alpha w} dw &= \int_{-\infty}^{-x} K(x, y)e^{i\alpha y} dy \cdot e^{i\alpha x} . \end{aligned}$$

The identity (\*) can be derived as follows:

$$(1-x)^b = \sum_{s=0}^b (-1)^s \binom{b}{s} x^s , \quad (1-x)^{-a} = \sum_{r=0}^\infty \binom{a+r-1}{a-1} x^r$$

and

$$(1-x)^{b-a} = \sum_{t=0}^{b-a} (-1)^t \binom{b-a}{t} x^t \quad (\text{we assume } b \geq a, \, b, \, a \text{ are natural numbers})$$

It follows that by equating the  $t^{\text{th}}$  power of  $(1-x)^{b-a}$  and the product of  $(1-x)^b$  and  $(1-x)^{-a}$ , we obtain

$$(-1)^t \binom{b-a}{t} = \sum_{s+r=t} (-1)^s \binom{b}{s} \binom{a+r-1}{a-1} = \sum_{s=0}^t (-1)^s \binom{b}{s} \binom{a-1+t-s}{a-1} ,$$

putting  $a = k+1$ ,  $b = \ell+1$ ,  $t = \ell-k$ , we obtain

$$(**) \quad (-1)^{\ell-k} = \sum_{s=0}^{\ell-k} (-1)^s \binom{\ell+1}{s} \binom{\ell-s}{k}.$$

And since

$$\begin{aligned} \binom{\ell+1}{s} \binom{\ell-s}{k} &= \frac{(\ell+1)!}{(s+1)! s!} \frac{(\ell-s)!}{(\ell-s-k)! k!} = \frac{\ell+1}{\ell-s+1} \frac{\ell!}{k! (\ell-k)!} \frac{(\ell-k)!}{(\ell-s-k)! s!} \\ &= (\ell+1) \binom{\ell}{k} \frac{\binom{\ell-k}{s}}{\ell-s+1}, \end{aligned}$$

by substituting the righthand side of the above into (\*\*) and dividing by  $(-1)^{\ell-k}$ , we obtain the identity (\*) easily.

To prove  $C_{m,n} = 0$  for  $m+n = \text{even}$ , if  $\gamma_{2m} = 0$ :

From (18), it is easily shown that  $C_{m,n}$  has at most the following terms:

$$\gamma_{m+n}, \gamma_{m+n-1}\gamma_0, \gamma_{m+n-2}\gamma_1, \gamma_{m+n-2}\gamma_0^2, \dots$$

These are isomorphic to all partitions of  $m+n+1$  into natural numbers such that  $\ell$  corresponds to  $\gamma_{\ell-1}$ ; e.g.,  $(m+n+1) \leftrightarrow \gamma_{m+n} (m+n-1, 1, 1) \leftrightarrow \gamma_{m+n-2}\gamma_0^2$  etc. Therefore, if  $m+n$  is even, then  $m+n+1$  is odd; hence all their partitions are of the form  $(p_1, p_2, \dots, p_i)$  such that one of  $p_i$  ( $1 \leq i \leq m+n+1$ ) is odd. Otherwise the sum  $\sum p_i$  is not an odd number since sums of even numbers are even. It follows that  $C_{m,n} = 0$  (if  $m+n = \text{even}$ ) since each of their terms has at least one factor  $\gamma_{p_i-1} = 0$  which corresponds to odd  $p_i$ .

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